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THERMAL EFFECTS ON THE RESPONSE OF CROSS-PLY LAMINATED SHALLOW SHELLS

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Abstract—The static response of cross-ply laminated shallow shells subjected to thermal loadings is investigated. An exact analytical solution using the state space approach is presented in conjunction with the Lévy method, for doubly curved, cylindrical and spherical shells under various boundary conditions. Numerical results of the higher-order theory of Reddy and Liu (1985, 1987) for center deflection of cross-ply laminated shallow shells are compared with those obtained using classical and first-order shell theories.

INTRODUCTION

The increased use of composite materials in aerospace and mechanical engineering structures is due to their high stiffness- and strength-to-weight ratio and to their anisotropic material property. Studies involving the thermoelastic behavior of composite plates and shells have been receiving greater attention in recent years [see Pell (1946), Stavsky (1963), Reddy and Hsu (1980), Wu and Tauchert (1980a,b), Kalam and Tauchert (1978), Avery and Herakovich (1986), Hyer and Cooper (1986) and Kardomateas (1989)]. The available results for the thermoelastic bending of laminates allow one to infer that the closed-form solutions involving various static problems were developed mainly for simply-supported edge conditions, and approximate methods were used for other boundary conditions. In this connection, a technique allowing one to obtain closed-form solutions for other boundary conditions is needed.

The objective of the present study is to investigate the thermal response behavior of laminated, cross-ply, composite shell panels using the third-order shell theory and to compare the results with those obtained using the classical and first-order shell theories. Analytical solutions of the theories are obtained using the state-space technique in conjunction with the Lévy method, allowing one to analyze the problems for a variety of boundary conditions. The exact solutions are presented to show the effects of variations in geometry, shallowness, lamination parameters and boundary conditions and the shear deformation on the thermal response of statically loaded layered anisotropic composite shell panels.

GOVERNING EQUATIONS

The third order theory (HSDT) used in the present study is based on the following displacement field [see Reddy and Liu (1985, 1987)]:

$$\bar{u} = \left(1+\frac{\zeta}{R_1}\right)u+\zeta\phi_1+\zeta^3\frac{4}{3h^2}\left(-\phi_1-\frac{1}{\gamma_1}\frac{\partial w}{\partial \xi_1}\right)$$

$$\bar{v} = \left(1+\frac{\zeta}{R_2}\right)v+\zeta\phi_2+\zeta^3\frac{4}{3h^2}\left(-\phi_2-\frac{1}{\gamma_2}\frac{\partial \omega}{\partial \xi_2}\right)$$

$$\bar{w} = w \qquad (1)$$

where $(\bar{u}, \bar{v}, \bar{w})$ are the displacements along the orthogonal curvilinear coordinates such that the ξ_1 and ξ_2-curves are lines of principal curvature on the midsurface $\zeta = 0$, and the ζ-curves are straight lines perpendicular to the surface $\zeta = 0$, (u, v, w) are the displacements of a point on the middle surface, and ϕ_1 and ϕ_2 are the rotations at $\zeta = 0$ of normals to the mid-surface with respect to the ξ_2 and ξ_1-axes, respectively. The parameters R_1 and R_2 denote the values of the principal radii of curvature of the middle surface, and γ_1 and γ_2 are the surface metrics defined in Reddy (1984). All displacement components $(u, v, w, \phi_1, \phi_2)$ are functions of (ξ_1, ξ_2).

Substituting eqn (1) into the strain–displacement relations of a shell referred to an orthogonal curvilinear coordinate system, we obtain:

$$\begin{aligned}
\varepsilon_1 &= \varepsilon_1^0 + \zeta(\kappa_1^0 + \zeta^2 \kappa_1^2) \\
\varepsilon_2 &= \varepsilon_2^0 + \zeta(\kappa_2^0 + \zeta^2 \kappa_2^2) \\
\varepsilon_4 &= \varepsilon_4^0 + \zeta^2 \kappa_4^1 \\
\varepsilon_5 &= \varepsilon_5^0 + \zeta^2 \kappa_5^1 \\
\varepsilon_6 &= \varepsilon_6^0 + \zeta(\kappa_6^0 + \zeta^2 \kappa_6^2)
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
&\varepsilon_1^0 = \frac{\partial u}{\partial x_1} + \frac{w}{R_1}, \quad \kappa_1^0 = \frac{\partial \phi_1}{\partial x_1}, \quad \kappa_1^2 = -c_2\left(\frac{\partial \phi_1}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2}\right) \\
&\varepsilon_2^0 = \frac{\partial v}{\partial x_2} + \frac{w}{R_2}, \quad \kappa_2^0 = \frac{\partial \phi_2}{\partial x_2}, \quad \kappa_2^2 = -c_2\left(\frac{\partial \phi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\right) \\
&\varepsilon_4^0 = \phi_2 + \frac{\partial w}{\partial x_2}, \quad \kappa_4^1 = -c_1\left(\phi_2 + \frac{\partial w}{\partial x_2}\right) \\
&\varepsilon_5^0 = \phi_1 + \frac{\partial w}{\partial x_1}, \quad \kappa_5^1 = -c_1\left(\phi_1 + \frac{\partial w}{\partial x_1}\right) \\
&\varepsilon_6^0 = \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}, \quad \kappa_6^0 = \frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} \\
&\kappa_6^2 = -c_2\left(\frac{\partial \phi_2}{\partial x_1} + \frac{\partial \phi_1}{\partial x_2} + 2\frac{\partial^2 w}{\partial x_1 \partial x_2}\right).
\end{aligned} \tag{3}$$

Here, x_i denote the Cartesian coordinates $(dx_i = \gamma_i \, d\xi_i, i = 1, 2)$, and $c_1 = 4/h^2$ and $c_2 = c_1/3$.

The stress-strain relations for the kth lamina are given by:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix}_{(k)} = \begin{pmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & 0 \\ & Q_{22}^{(k)} & 0 & 0 & 0 \\ & & Q_{66}^{(k)} & 0 & 0 \\ & & & Q_{44}^{(k)} & 0 \\ \text{Symm.} & & & & Q_{55}^{(k)} \end{pmatrix} \begin{Bmatrix} \varepsilon_1 - \alpha_{11}^{(k)}\Delta T \\ \varepsilon_2 - \alpha_{22}^{(k)}\Delta T \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{4}$$

where $Q_{ij}^{(k)}$ are the material coefficients of the kth lamina in the laminate coordinate system and $\alpha_{11}^{(k)}$ and $\alpha_{22}^{(k)}$ are the coefficients of linear thermal expansion for layer k in the laminate coordinates; ΔT denotes the temperature rise in the laminate and is given by:

$$\Delta T = T_0(x_1, x_2) + \zeta T_1(x_1, x_2). \tag{5}$$

Using Hamilton's principle, the governing equations appropriate for the displacement field

(1) and the constitutive equation (4) are derived in Reddy and Liu (1985) as:

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=0$$

$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=0$$

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}-c_1\left(\frac{\partial K_1}{\partial x_1}+\frac{\partial K_2}{\partial x_2}\right)+c_2\left(\frac{\partial^2 P_1}{\partial x_1^2}+\frac{\partial^2 P_2}{\partial x_2^2}+2\frac{\partial^2 P_6}{\partial x_1\,\partial x_2}\right)-\frac{N_1}{R_1}-\frac{N_2}{R_2}+q=0$$

$$\frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2}-Q_1+c_1K_1-c_2\left(\frac{\partial P_1}{\partial x_1}+\frac{\partial P_6}{\partial x_2}\right)=0$$

$$\frac{\partial M_6}{\partial x_1}+\frac{\partial M_2}{\partial x_2}-Q_2+c_1K_2-c_2\left(\frac{\partial P_6}{\partial x_1}+\frac{\partial P_2}{\partial x_2}\right)=0 \tag{6}$$

where q is the distributed transverse mechanical load, and N_i and M_i, etc., are the stress resultants,

$$(N_i, M_i, P_i)=\sum_{k=1}^{N}\int_{\zeta^{k-1}}^{\zeta^k}\sigma_i^{(k)}(1,\zeta,\zeta^3)\,\mathrm{d}\zeta \quad (i=1,2,6),$$

$$(Q_1, K_1)=\sum_{k=1}^{N}\int_{\zeta^{k-1}}^{\zeta^k}\sigma_5^{(k)}(1,\zeta^2)\,\mathrm{d}\zeta$$

$$(Q_2, K_2)=\sum_{k=1}^{N}\int_{\zeta^{k-1}}^{\zeta^k}\sigma_4^{(k)}(1,\zeta^2)\,\mathrm{d}\zeta. \tag{7}$$

The resultants are related to the total strains by the equations

$$\begin{aligned} N_i &= A_{ij}\varepsilon_j^0+B_{ij}\kappa_j^0+E_{ij}\kappa_j^2-N_i^T\\ M_i &= B_{ij}\varepsilon_j^0+D_{ij}\kappa_j^0+F_{ij}\kappa_j^2-M_i^T, \quad (i,j=1,2,6)\\ P_i &= E_{ij}\varepsilon_j^0+F_{ij}\kappa_j^0+H_{ij}\kappa_j^2-P_i^T \end{aligned} \tag{8}$$

$$\begin{aligned} Q_2 &= A_{4j}\varepsilon_j^0+D_{4j}\kappa_j^1\\ Q_1 &= A_{5j}\varepsilon_j^0+D_{5j}\kappa_j^1\\ K_2 &= D_{4j}\varepsilon_j^0+F_{4j}\kappa_j^1, \quad (j=4,5)\\ K_1 &= D_{5j}\varepsilon_j^0+F_{5j}\kappa_j^1 \end{aligned} \tag{9}$$

where A_{ij}, B_{ij}, etc., are the laminate stiffnesses,

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij})=\sum_{k=1}^{N}\int_{\zeta^{k-1}}^{\zeta^k}Q_{ij}^{(k)}(1,\zeta,\zeta^2,\zeta^3,\zeta^4,\zeta^6,)\,\mathrm{d}\zeta,$$

$$\text{for}\quad i,j=1,2,4,5,6, \tag{10}$$

and the thermal forces and moments are defined by

$$\begin{Bmatrix} N_1^T, & M_1^T, & P_1^T\\ N_2^T, & M_2^T, & P_2^T \end{Bmatrix}=\sum_{k=1}^{N}\int_{\zeta^{k-1}}^{\zeta^k}\begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)}\\ Q_{12}^{(k)} & Q_{22}^{(k)} \end{bmatrix}\begin{Bmatrix} \alpha_{11}^{(k)}\\ \alpha_{22}^{(k)} \end{Bmatrix}(1,\zeta,\zeta^3)\Delta T\,\mathrm{d}\zeta. \tag{11}$$

For the sake of completeness and comparison, the governing equations of the classical (CST) and the first-order (FSDT) shell theories are also presented.

1. *Classical theory* (CST)

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=0$$
$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=0$$
$$\frac{\partial^2 M_1}{\partial x_1^2}+\frac{\partial^2 M_6}{\partial x_1\,\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}-\frac{N_1}{R_1}-\frac{N_2}{R_2}+q=0. \tag{12}$$

The resultants N_i and M_i are given in eqns (7) and (8) with $E_{ij}=F_{ij}=0$, where the displacement functions ϕ_1 and ϕ_2 in this case are to be replaced by the expressions:

$$\phi_i=-\frac{\partial w}{\partial x_i}\quad (i=1,2).$$

2. *First-order theory* (FSDT)

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=0$$
$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=0$$
$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}-\frac{N_1}{R_1}-\frac{N_2}{R_2}+q=0$$
$$\frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2}-Q_1=0$$
$$\frac{\partial M_6}{\partial x_1}+\frac{\partial M_2}{\partial x_2}-Q_2=0. \tag{13}$$

The resultants (N_i, M_i) can be expressed in terms of the strains as in eqns (8) with $E_{ij}=F_{ij}=0$. The resultant shear forces Q_1 and Q_2 are given by:

$$Q_2=K_4^2A_{44}\left(\phi_2+\frac{\partial w}{\partial x_2}-\frac{v}{R_2}\right)$$
$$Q_1=K_5^2A_{55}\left(\phi_1+\frac{\partial w}{\partial x_1}-\frac{u}{R_1}\right) \tag{14}$$

where K_4^2 and K_5^2 are the shear correction factors.

SOLUTION PROCEDURE

A generalized Lévy type solution, in conjunction with the state space approach is used to analyze the thermal bending of cross-ply laminated shallow shells. The edges $x_2=0$, b are assumed to be simply supported, while the remaining ones ($x_1=\pm a/2$) may have arbitrary combinations of free, clamped and simply-supported edge conditions. We express the generalized displacements as products of undetermined functions and known trigonometric functions so as to identically satisfy the simply-supported boundary conditions at $x_2 = 0, b$:

$$u = w = \phi_1 = N_2 = M_2 = P_2 = 0 \quad \text{for} \quad \text{HSDT}$$
$$u = w = \phi_1 = N_2 = M_2 = 0 \quad \text{for} \quad \text{FSDT}$$
$$u = w = N_2 = M_2 = 0 \quad \text{for} \quad \text{CST.} \tag{15}$$

A sinusoidal distribution of the thermal loadings will be considered, which in the present case takes the form:

$$\begin{Bmatrix} T_0 \\ T_1 \end{Bmatrix} = \begin{Bmatrix} \bar{T}_0 \\ \bar{T}_1 \end{Bmatrix} \cos \alpha x_1 \sin \beta x_2 \tag{16}$$

where $\alpha = \pi/\alpha$ and $\beta = \pi/b$ in all the numerical results and the mechanical loading q is considered to be zero throughout the analysis.

The displacement quantities will be represented as:

$$\begin{Bmatrix} u(x_1, x_2) \\ v(x_1, x_2) \\ w(x_1, x_2) \\ \phi_1(x_1, x_2) \\ \phi_2(x_1, x_2) \end{Bmatrix} = \begin{Bmatrix} U(x_1) \sin \beta x_2 \\ V(x_1) \cos \beta x_2 \\ W(x_1) \sin \beta x_2 \\ X(x_1) \sin \beta x_2 \\ Y(x_1) \cos \beta x_2 \end{Bmatrix}. \tag{17}$$

The representation (17) is valid for HSDT, FSDT and CST. Substitution of eqn (17) into the governing equations of the three theories, we obtain five differential equations for HSDT and FSDT and three differential equations for CLT. In order to represent the system of differential equations in the form needed for the state-space approach, the following variables are introduced:

HSDT

$$Z_1 = U, \quad Z_2 = U', \quad Z_3 = V, \quad Z_4 = V', \quad Z_5 = W, \quad Z_6 = W',$$
$$Z_7 = W'', \quad Z_8 = W''', \quad Z_9 = X, \quad Z_{10} = X', \quad Z_{11} = Y, \quad Z_{12} = Y'; \tag{18}$$

FSDT

$$Z_1 = U, \quad Z_2 = U', \quad Z_3 = V, \quad Z_4 = V', \quad Z_5 = W, \quad Z_6 = W',$$
$$Z_7 = X, \quad Z_8 = X', \quad Z_9 = Y, \quad Z_{10} = Y'; \tag{19}$$

CST

$$Z_1 = U, \quad Z_2 = U', \quad Z_3 = V, \quad Z_4 = V',$$
$$Z_5 = W, \quad Z_6 = W', \quad Z_7 = W'', \quad Z_8 = W''', \tag{20}$$

where the primes over the variables indicate differentiation with respect to x_1. The differential equations take the form:

$$\mathbf{Z}' = \mathbf{B}Z + \mathbf{r} \tag{21}$$

where the matrix $[B]$ is defined in Appendix I for HSDT, FSDT and CST. The load vector r is defined as:

HSDT

$$\mathbf{r} = \{0, g_1 \sin \alpha x_1, 0, g_2 \cos \alpha x_1, 0, 0, 0, g_3 \cos \alpha x_1, 0, g_4 \sin \alpha x_1, 0, g_5 \cos \alpha x_1\}^T \tag{22}$$

FSDT

$$\mathbf{r} = \{0, g_1 \sin \alpha x_1, 0, g_2 \cos \alpha x_1, 0, g_3 \cos \alpha x_1, 0, g_4 \sin \alpha x_1, 0, g_5 \cos \alpha x_1\}^T \tag{23}$$

CST

$$\mathbf{r} = \{0, g_1 \sin \alpha x_1, 0, g_2 \cos \alpha x_1, 0, 0, 0, g_3 \cos \alpha x_1\}^T \tag{24}$$

where the coefficients $g_1, g_2, \ldots, g_5$ are defined in Appendix II for the three theories.

The solution to eqn (21) is [see Reddy and Khdeir (1989, 1990)]

$$\mathbf{Z} = e^{Bx_1}\left\{\mathbf{K} + \int e^{-B\eta}\, \mathbf{r}\, d\eta\right\}. \tag{25}$$

Here K is constant column vector to be determined from the edge conditions while e^{Bx_1} is expressed as:

$$e^{Bx_1} = [S]\begin{bmatrix} e^{\lambda_1 x_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n x_1} \end{bmatrix}[S]^{-1}, \tag{26}$$

where, $n = 12$ for HSDT, $n = 10$ for FSDT and $n = 8$ for CST. λ_i denote the distinct eigenvalues of $[B]$, while $[S]$ denotes the matrix of eigenvectors of $[B]$.

The boundary conditions for simply-supported (S), clamped (C) and free (F) at the edges $x_1 = \pm a/2$ for the three theories are:

HSDT

$$\begin{aligned} S&: v = w = \phi_2 = N_1 = M_1 = P_1 = 0 \\ C&: u = v = w = \frac{\partial w}{\partial x_1} = \phi_1 = \phi_2 = 0 \\ F&: N_1 = M_1 = P_1 = N_6 = M_6 - c_2 P_6 = 0 \\ &Q_1 - c_1 K_1 + c_2\left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_6}{\partial x_2}\right) = 0 \end{aligned} \tag{27}$$

FSDT

$$\begin{aligned} S&: v = w = \phi_2 = N_1 = M_1 = 0 \\ C&: u = v = w = \phi_1 = \phi_2 = 0 \\ F&: N_1 = M_1 = Q_1 = N_6 = M_6 = 0 \end{aligned} \tag{28}$$

CST

$$\begin{aligned} S&: v = w = N_1 = M_1 = 0 \\ C&: u = v = w = \frac{\partial w}{\partial x_1} = 0 \\ F&: N_1 = M_1 = N_6 = 0 \\ &\frac{\partial M_1}{\partial x_1} + 2\frac{\partial M_6}{\partial x_2} = 0. \end{aligned} \tag{29}$$

NUMERICAL RESULTS AND CONCLUSIONS

Numerical results are displayed to obtain the trend of variation in the thermal response with the variation of geometry, lamination and boundary conditions. The non-dimensionalized center deflection of cross-ply cylindrical, spherical and doubly-curved panels for the lamination schemes (0/90), (0/90 0) and (0/90/... 10 layers) have been displayed, for various combinations of boundary conditions in Tables 1–3 and Figs 1–6. It was assumed that the thickness and the material for all the laminae are the same, having the following characteristics:

$$E_1 = 25E_2,\quad G_{12} = G_{13} = 0.5E_2,\quad G_{23} = 0.2E_2,\quad \nu_{12} = 0.25,\quad \alpha_2/\alpha_1 = 3.$$

Table 1. Non-dimensionalized center deflections $\bar{w}$ of (0/90) cylindrical shells subjected to sinusoidal temperature distributed load for various boundary conditions ($a/b = 1$, $a/h = 10$, $R_1 = \infty$, $R_2 = R$)

R/a	Theory	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC
	HSDT	1.1235	0.7441	0.5158	1.2617	1.2001	0.7750
5	FSDT	1.1248	0.7551	0.5297	1.2657	1.2028	0.7858
	CST	1.1280	0.7091	0.4703	1.2550	1.2013	0.7446
	HSDT	1.1421	0.7544	0.5177	1.2680	1.2112	0.7760
10	FSDT	1.1439	0.7658	0.5319	1.2722	1.2142	0.7871
	CST	1.1447	0.7161	0.4702	1.2616	1.2117	0.7435
	HSDT	1.1482	0.7583	0.5169	1.2694	1.2145	0.7738
50	FSDT	1.1501	0.7699	0.5312	1.2738	1.2176	0.7851
	CST	1.1501	0.7183	0.4687	1.2638	1.2151	0.7403
	HSDT	1.1485	0.7586	0.5164	1.2693	1.2145	0.7728
Plate	FSDT	1.1504	0.7703	0.5307	1.2736	1.2176	0.7842
	CST	1.1504	0.7183	0.4681	1.2639	1.2152	0.7392

Table 2. Non-dimensionalized center deflections $\bar{w}$ of (0/90) spherical shells subjected to sinusoidal temperature distributed load for various boundary conditions ($a/b = 1$, $a/h = 10$)

R/a	Theory	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC
	HSDT	1.0545	0.6737	0.2148	1.1965	1.1310	0.7101
5	FSDT	1.0546	0.6808	0.2097	1.1987	1.1322	0.7173
	CST	1.0660	0.6549	0.2540	1.1909	1.1366	0.6918
	HSDT	1.1235	0.7287	0.3677	1.2524	1.1935	0.7518
10	FSDT	1.1248	0.7388	0.3711	1.2561	1.1960	0.7617
	CST	1.1280	0.6965	0.3666	1.2449	1.1946	0.7236
	HSDT	1.1475	0.7550	0.4897	1.2693	1.2140	0.7705
50	FSDT	1.1493	0.7665	0.5020	1.2736	1.2171	0.7817
	CST	1.1494	0.7157	0.4499	1.2632	1.2144	0.7375
	HSDT	1.1485	0.7586	0.5164	1.2693	1.2145	0.7728
Plate	FSDT	1.1504	0.7703	0.5307	1.2736	1.2176	0.7842
	CST	1.1504	0.7183	0.4681	1.2639	1.2152	0.7392

Table 3. Non-dimensionalized center deflections $\bar{w}$ of ten-layer (0/90/...) cylindrical shells subjected to sinusoidal temperature distributed load for various boundary conditions ($a/b = 1$, $a/h = 10$, $R_1 = \infty$, $R_2 = R$)

R/a	Theory	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC
	HSDT	1.0216	0.7032	0.4849	1.0666	1.0480	0.6854
5	FSDT	1.0215	0.7087	0.4917	1.0670	1.0482	0.6909
	CST	1.0247	0.6189	0.3905	1.0643	1.0492	0.6206
	HSDT	1.0303	0.7082	0.4874	1.0705	1.0537	0.6884
10	FSDT	1.0302	0.7138	0.4942	1.0709	1.0539	0.6940
	CST	1.0310	0.6214	0.3913	1.0671	1.0533	0.6218
	HSDT	1.0332	0.7099	0.4880	1.0718	1.0555	0.6892
50	FSDT	1.0330	0.7156	0.4950	1.0722	1.0557	0.6949
	CST	1.0331	0.6222	0.3914	1.0680	1.0546	0.6219
	HSDT	1.0333	0.7101	0.4880	1.0718	1.0556	0.6892
Plate	FSDT	1.0331	0.7157	0.4949	1.0722	1.0558	0.6949
	CST	1.0331	0.6222	0.3914	1.0681	1.0546	0.6219

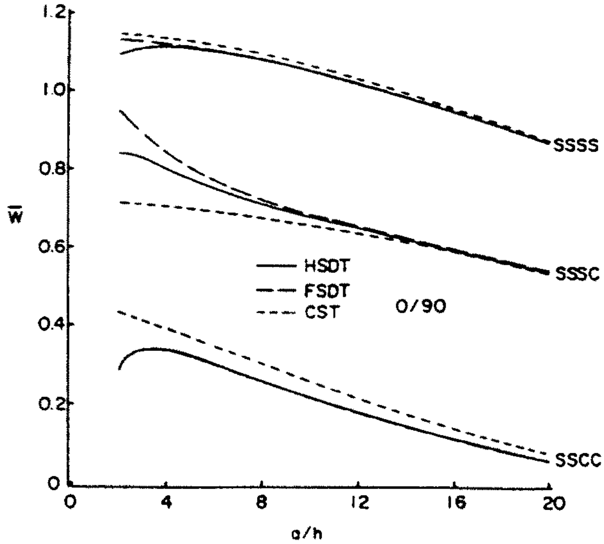

Fig. 1. Non-dimensionalized center deflection versus side to thickness ratio of (0/90) spherical shells subjected to sinusoidal temperature distributed load for SSSS, SSSC and SSCC boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

The shear correction factors ($K_4^2 = K_5^2$) for the first-order shear deformation shell theory (FSDT) are taken to be 5/6. The following non-dimensionalized deflection parameter has been used throughout the calculations:

$$\bar{w} = w(0, b/2)\frac{10}{\alpha_1 T_1 b^2},$$

where T_0 and q are considered to be zero. The notation SSFC, for example, means that the edges $x_2 = 0, b$ are simply supported, $x_1 = -a/2$ is clamped and $x_1 = 1/2$ is free. In addition to the effects played by the boundary conditions on the thermal response, the numerical

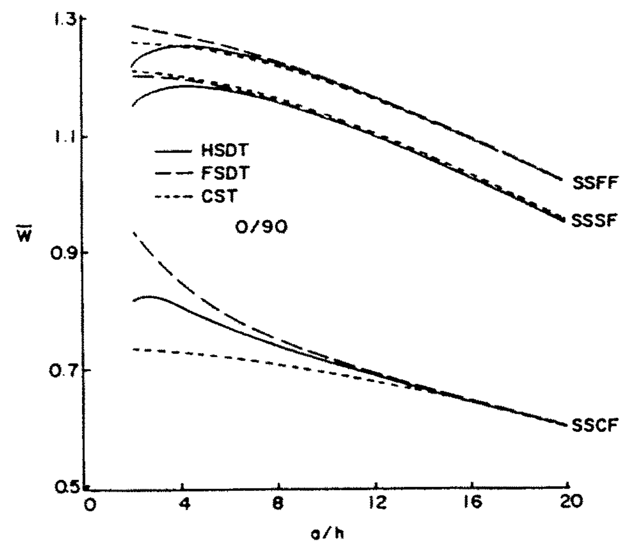

Fig. 2. Non-dimensionalized center deflection versus side to thickness ratio of (0/90) spherical shells subjected to sinusoidal temperature distributed load for SSSF, SSCF and SSFF boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

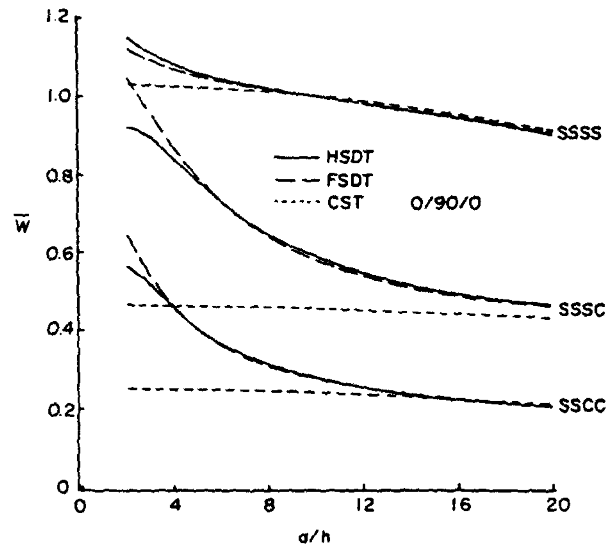

Fig. 3. Non-dimensionalized center deflection versus side to thickness ratio of (0/90/0) spherical shells subjected to sinusoidal temperature distributed load for SSSS, SSSC and SSCC boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

results allow one to conclude the following:

(1) For thick panels the effect of transverse shear deformation is always to be incorporated into the analysis, because CST underpredicts the panel response when compared to FSDT and HSDT. An exception to this observation is provided by the SSSS boundary conditions. The deflections predicted by the classical shell theory differ by about 6% at the most.

(2) For all lamination schemes, the deflection w of cylindrical panels is higher than the spherical ones.

(3) For moderately thick panels, the results predicted by HSDT and FSDT are in excellent agreement.

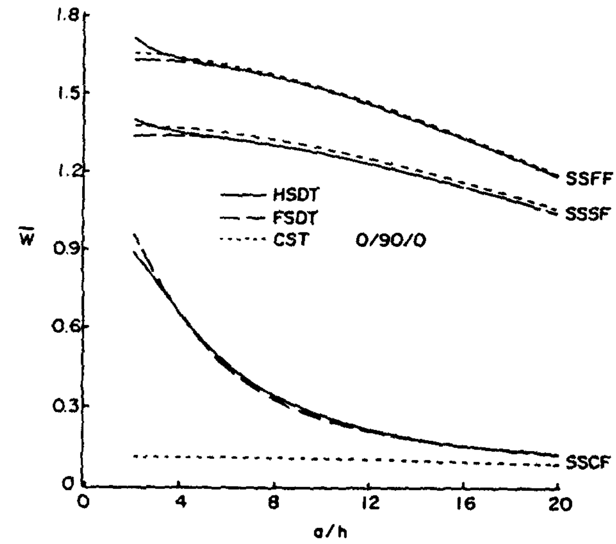

Fig. 4. Non-dimensionalized center deflection versus side to thickness ratio of (0/90/0) spherical shells subjected to sinusoidal temperature distributed load for SSSF, SSCF and SSFF boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

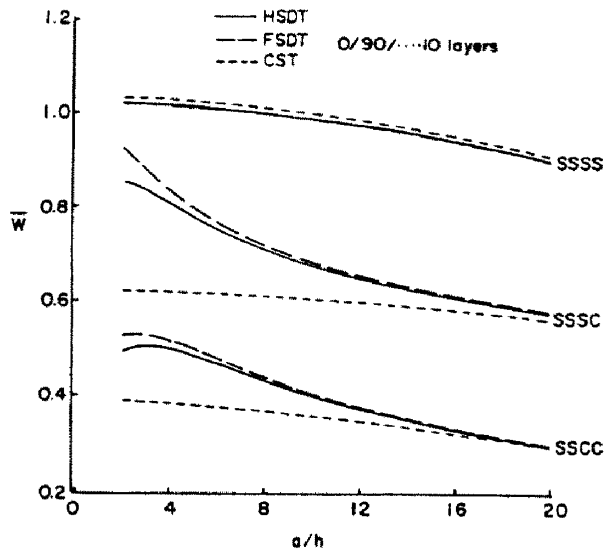

Fig. 5. Non-dimensionalized center deflection versus side to thickness ratio of ten-layer (0/90/...) spherical shells subjected to sinusoidal temperature distributed load for SSSS, SSSC and SSCC boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

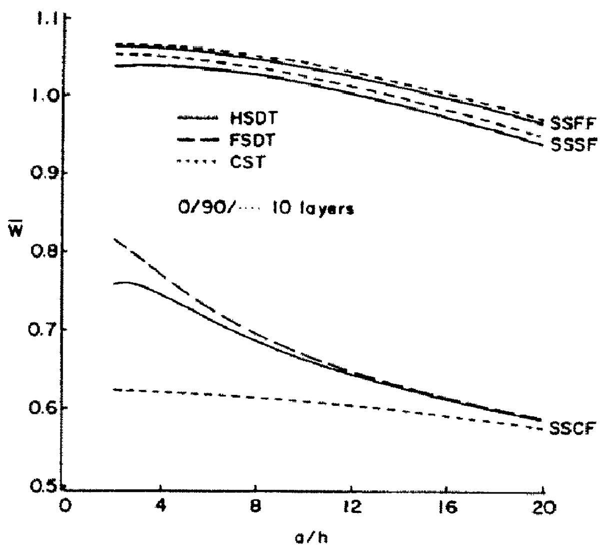

Fig. 6. Non-dimensionalized center deflection versus side to thickness ratio of ten-layer (0/90/...) spherical shells subjected to sinusoidal temperature distributed load for SSSF, SSCF and SSFF boundary conditions ($a/b = 1$, $R_1 = R_2 = 5a$).

(4) The mathematical tool, namely the state space approach, used to provide exact solutions for the static thermal response problems of laminated composite shallow shells for various boundary conditions has been found to be of great computational efficiency and has not attained so far for this particular case.

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APPENDIX I

The matrix $[B]$ *coefficients*

HSDT

$$[B] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1 & 0 & 0 & b_2 & 0 & b_3 & 0 & b_4 & b_5 & 0 & 0 & b_6 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b_7 & b_8 & 0 & b_9 & 0 & b_{10} & 0 & 0 & b_{11} & b_{12} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & b_{13} & b_{14} & 0 & b_{15} & 0 & b_{16} & 0 & 0 & b_{17} & b_{18} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
b_{19} & 0 & 0 & b_{20} & 0 & b_{21} & 0 & b_{22} & b_{23} & 0 & 0 & b_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & b_{25} & b_{26} & 0 & b_{27} & 0 & b_{28} & 0 & 0 & b_{29} & b_{30} & 0
\end{bmatrix}$$

where:

$$b_1 = (e_7 e_{30} - e_3 e_{34})/e_0, \quad b_2 = (e_2 e_{30} - e_3 e_{29})/e_0$$
$$b_3 = (e_6 e_{30} - e_3 e_{33})/e_0, \quad b_4 = (e_5 e_{30} - e_3 e_{32})/e_0$$
$$b_5 = (e_8 e_{30} - e_3 e_{35})/e_0, \quad b_6 = (e_4 e_{30} - e_3 e_{31})/e_0$$
$$b_7 = (e_9 e_{39} - e_{12} e_{36})/c_0, \quad b_8 = (e_{14} e_{39} - e_{12} e_{41})/e_0$$
$$b_9 = (e_{16} e_{39} - e_{12} e_{43})/c_0, \quad b_{10} = (e_{13} e_{39} - e_{12} e_{40})/c_0$$
$$b_{11} = (e_{11} e_{39} - e_{12} e_{38})/c_0, \quad b_{12} = (e_{15} e_{39} - e_{12} e_{42})/c_0$$
$$b_{19} = (e_1 e_{34} - e_7 e_{28})/e_0, \quad b_{20} = (e_1 e_{29} - e_2 e_{28})/e_0$$
$$b_{21} = (e_1 e_{33} - e_6 e_{28})/e_0, \quad b_{22} = (e_1 e_{32} - e_5 e_{28})/e_0$$
$$b_{23} = (e_1 e_{35} - e_8 e_{28})/c_0, \quad b_{24} = (e_1 e_{31} - e_4 e_{28})/e_0$$
$$b_{25} = (e_{10} e_{36} - e_9 e_{37})/c_0, \quad b_{26} = (e_{10} e_{41} - e_{14} e_{37})/c_0$$
$$b_{27} = (e_{10} e_{43} - e_{16} e_{37})/c_0, \quad b_{28} = (e_{10} e_{40} - e_{13} e_{37})/c_0$$
$$b_{29} = (e_{10} e_{38} - e_{11} e_{37})/c_0, \quad b_{30} = (e_{10} e_{42} - e_{15} e_{37})/c_0$$

$$b_{13} = a_0(b_1e_{21}+b_7a_1+b_{25}a_2+b_{19}e_{23}+e_{26})$$
$$b_{14} = a_0(b_8a_1+b_{26}a_2+e_{27}), \quad b_{15} = a_0(b_9a_1+b_{27}a_2+e_{20})$$
$$b_{16} = a_0(e_{18}+b_3e_{21}+b_{21}e_{23}+b_{10}a_1+b_{28}a_2)$$
$$b_{17} = a_0(e_{17}+b_5e_{21}+b_{23}e_{23}+b_{11}a_1+b_{29}a_2)$$
$$b_{18} = a_0(e_{19}+b_{12}a_1+b_{30}a_2)$$

$$e_0 = e_3e_{28}-e_1e_{30}, \quad c_0 = e_{12}e_{37}-e_{10}e_{39}$$
$$a_0 = -1/(b_4e_{21}+b_{22}e_{23}+e_{25})$$
$$a_1 = b_2e_{21}+b_{20}e_{23}+e_{22}, \quad a_2 = b_6e_{21}+b_{24}e_{23}+e_{24}$$
$$e_1 = A_{11}, \quad e_2 = -\beta(A_{12}+A_{60})$$
$$e_3 = B_{11}-c_2E_{11}, \quad e_4 = \beta[c_2(E_{12}+E_{66})-B_{12}-B_{66}]$$
$$e_5 = -c_2E_{11}, \quad e_6 = c_2[\beta^2(E_{12}+2E_{66})]+\frac{A_{11}}{R_1}+\frac{A_{12}}{R_2}$$
$$e_7 = -\beta^2A_{66}, \quad e_8 = \beta^2(c_2E_{66}-B_{66})$$
$$e_9 = -e_2, \quad e_{10} = A_{66}$$
$$e_{11} = -e_4, \quad e_{12} = B_{66}-c_2E_{66}$$
$$e_{13} = -c_2\beta(E_{12}+2E_{66}), \quad e_{14} = -\beta^2A_{22}$$
$$e_{15} = \beta^2(c_2E_{22}-B_{22}), \quad e_{16} = c_2\beta^3E_{22}+\beta\left(\frac{A_{12}}{R_1}+\frac{A_{22}}{R_2}\right)$$
$$e_{17} = A_{55}-c_1D_{55}-c_1(D_{55}-c_1F_{55})+c_2\beta^2[c_2(H_{12}-2H_{66})-(F_{12}+2F_{66})] - \frac{1}{R_1}(B_{11}-c_2E_{11})-\frac{1}{R_2}(B_{12}-c_2E_{12})$$
$$e_{18} = A_{55}-c_1D_{55}-c_1(D_{55}-c_1F_{55})+c_2^2\beta^2[2H_{12}+4H_{66}]+2c_2\left(\frac{E_{11}}{R_1}+\frac{E_{12}}{R_2}\right)$$
$$e_{19} = -\beta[A_{44}-c_1D_{44}-c_1(D_{44}-c_1F_{44})]+c_2\beta^3(F_{22}-c_2H_{22})+\frac{\beta}{R_1}(B_{12}-c_2E_{12})+\frac{\beta}{R_2}(B_{22}-c_2E_{22})$$
$$e_{20} = -\beta^2[A_{44}-c_1D_{44}-c_1(D_{44}-c_1F_{44})]-c_2^2\beta^4H_{22}-2c_2\beta^2\left(\frac{E_{12}}{R_1}+\frac{E_{22}}{R_2}\right)-\frac{A_{11}}{R_1^2}-\frac{2A_{12}}{R_1R_2}-\frac{A_{22}}{R_2^2}$$
$$e_{21} = -e_5, \quad e_{22} = e_{13}$$
$$e_{23} = c_2(F_{11}-c_2H_{11}),$$
$$e_{24} = c_2\beta[c_2(2H_{66}+H_{12})-(F_{12}+2F_{66})], \quad e_{25} = -c_2^2H_{11}$$
$$e_{26} = -e_6, \quad e_{27} = e_{16}, \quad e_{28} = e_3, \quad e_{29} = e_4$$
$$e_{30} = D_{11}-2c_2F_{11}+c_2^2H_{11}$$
$$e_{31} = \beta[2c_2(F_{12}+F_{66})-c_2^2(H_{12}+H_{66})-D_{12}-D_{66}]$$
$$e_{32} = -e_{23}, \quad e_{33} = -e_{17}, \quad e_{34} = e_8$$
$$e_{35} = c_1(D_{55}-c_1F_{55})-(A_{55}-c_1D_{55})+\beta^2[2c_2F_{66}-D_{66}-c_2^2H_{66}]$$
$$e_{36} = -e_4, \quad e_{37} = e_{12}, \quad e_{38} = -e_{31}$$
$$e_{39} = D_{66}-2c_2F_{66}+c_2^2H_{66}$$
$$e_{40} = e_{24}, \quad e_{41} = e_{15},$$
$$e_{42} = c_1(D_{44}-c_1F_{44})-(A_{44}-c_1D_{44})+\beta^2[2c_2F_{22}-D_{22}-c_2^2H_{22}], \quad e_{43} = e_{19}.$$

FSDT

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & b_2 & 0 & b_3 & b_4 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_6 & b_7 & 0 & b_8 & 0 & 0 & b_9 & b_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & 0 & b_{13} & 0 & 0 & b_{14} & b_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ b_{16} & 0 & 0 & b_{17} & 0 & b_{18} & b_{19} & 0 & 0 & b_{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b_{21} & b_{22} & 0 & b_{23} & 0 & 0 & b_{24} & b_{25} & 0 \end{bmatrix}$$

where:

$$b_1 = (e_3e_{21}-e_5e_{19})/e_0, \quad b_2 = (e_3e_{18}-e_2e_{19})/e_0$$
$$b_3 = (e_3e_{23}-e_{19}e_{31})/e_0, \quad b_4 = (e_3e_{22}-e_6e_{19})/e_0$$
$$b_5 = (e_3e_{20}-e_4e_{19})/e_0, \quad b_6 = (e_8e_{27}-e_{10}e_{25})/C_0$$
$$b_7 = (e_{11}e_{27}-e_{10}e_{28})/C_0, \quad b_8 = (e_{27}e_{32}-e_{10}e_{30})/C_0$$
$$b_9 = (e_9e_{27}-e_{10}e_{26})/C_0, \quad b_{10} = (e_{12}e_{27}-e_{10}e_{29})/C_0$$
$$b_{11} = -e_{33}/e_{13}, \quad b_{12} = -e_{34}/e_{13}, \quad b_{13} = -e_{15}/e_{13},$$
$$b_{14} = -e_{14}/e_{13}, \quad b_{15} = -e_{16}/e_{13}, \quad b_{16} = (e_5e_{17}-e_1e_{21})/e_0,$$
$$b_{17} = (e_2e_{17}-e_1e_{18})/e_0, \quad b_{18} = (e_{17}e_{31}-e_1e_{23})/e_0$$
$$b_{19} = (e_6e_{17}-e_1e_{22})/e_0, \quad b_{20} = (e_4e_{17}-e_1e_{20})/e_0$$
$$b_{21} = (e_7e_{25}-e_8e_{24})/C_0, \quad b_{22} = (e_7e_{28}-e_{11}e_{24})/C_0$$
$$b_{23} = (e_7e_{30}-e_{24}e_{32})/C_0, \quad b_{24} = (e_7e_{26}-e_9e_{24})/C_0$$
$$b_{25} = (e_7e_{29}-e_{21}e_{24})/C_0, \quad e_0 = e_1e_{19}-e_3e_{17}$$
$$C_0 = e_{10}e_{24}-e_7e_{27}, \quad e_1 = A_{11}, \quad e_2 = -\beta(A_{12}+A_{66})$$
$$e_3 = B_{11}, \quad e_4 = -\beta(B_{12}+B_{66})$$
$$e_5 = -\beta^2A_{66}, \quad e_6 = -\beta^2B_{66}, \quad e_7 = A_{66}$$
$$e_8 = -e_2, \quad e_9 = -e_4, \quad e_{10} = B_{66}$$
$$e_{11} = -\beta^2A_{22}, \quad e_{12} = -\beta^2B_{22}, \quad e_{13} = K_5^2A_{55}$$
$$e_{14} = K_5^2A_{55} - \frac{B_{11}}{R_1} - \frac{B_{12}}{R_2}$$
$$e_{15} = -\beta^2K_4^2A_{44} - \frac{A_{11}}{R_1^2} - \frac{2A_{12}}{R_1R_2} - \frac{A_{22}}{R_2^2}$$
$$e_{16} = -\beta K_4^2A_{44} + \beta\left(\frac{B_{12}}{R_1} + \frac{B_{22}}{R_2}\right)$$
$$e_{17} = e_3, \quad e_{18} = e_4, \quad e_{19} = D_{11},$$
$$e_{20} = -\beta(D_{12}+D_{66}, \quad e_{21} = -\beta^2B_{66} + K_5^2\frac{A_{55}}{R_1}$$
$$e_{22} = -\beta^2D_{66} - K_5^2A_{55}, \quad e_{23} = -e_{14}$$
$$e_{24} = e_{10}, \quad e_{25} = e_9, \quad e_{26} = -e_{20}, \quad e_{27} = D_{66}$$
$$e_{28} = -\beta^2B_{22} + K_4^2\frac{A_{44}}{R_2}, \quad e_{29} = -\beta^2D_{22} - K_4^2A_{44}$$
$$e_{30} = e_{16}, \quad e_{31} = \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}$$
$$e_{32} = \beta\left(\frac{A_{12}}{R_1} + \frac{A_{22}}{R_2}\right), \quad e_{33} = -\left(\frac{A_{11}}{R_1} + \frac{A_{12}}{R_2} + K_5^2\frac{A_{55}}{R_1}\right)$$
$$e_{34} = \beta\left(\frac{A_{12}}{R_1} + \frac{A_{22}}{R_2} + K_4^2\frac{A_{44}}{R_2}\right)$$

CST

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & b_2 & 0 & b_3 & 0 & b_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b_5 & b_6 & 0 & b_7 & 0 & b_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b_9 & b_{10} & 0 & b_{11} & 0 & b_{12} & 0 \end{bmatrix}$$

where:

$$b_1 = -e_2/e_1, \quad b_2 = -e_3/e_1, \quad b_3 = -e_5/e_1, \quad b_4 = -e_4/e_1$$

$$b_5 = -e_6/e_7,\quad b_6 = -e_8/e_7,\quad b_7 = -e_{10}/e_7,\quad b_8 = -e_9/e_7$$

$$b_9 = -e_{21}/e_{18},\quad b_{10} = -e_{22}/e_{18},\quad b_{11} = -e_{20}/e_{18},\quad b_{12} = -e_{19}/e_{18}$$

and,

$$e_1 = A_{11},\quad e_2 = -\beta^2 A_{66},\quad e_3 = -\beta(A_{12}+A_{66}),\quad e_4 = -B_{11}$$

$$e_5 = \beta^2(B_{12}+2B_{66}) + \frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}$$

$$e_6 = -e_3,\quad e_7 = A_{66},\quad e_8 = -\beta^2 A_{22}$$

$$e_9 = -\beta(B_{12}+2B_{66}),\quad e_{10} = \beta^3 B_{22} + \beta\left(\frac{A_{12}}{R_1} + \frac{A_{22}}{R_2}\right)$$

$$e_{11} = D_{11},\quad e_{12} = -2\beta^2(D_{12}+2D_{66}) - 2\left(\frac{B_{11}}{R_1} + \frac{B_{12}}{R_2}\right)$$

$$e_{13} = \beta^4 D_{22} + \frac{A_{11}}{R_1^2} + 2\frac{A_{12}}{R_1 R_2} + \frac{A_{22}}{R_2^2} + 2\beta^2\frac{B_{12}}{R_1} + 2\beta^2\frac{B_{22}}{R_2}$$

$$e_{14} = e_4,\quad e_{15} = e_5$$

$$e_{16} = -e_9,\quad e_{17} = -e_{10},\quad e_{18} = e_{11} - e_4 e_{14}/e_1$$

$$e_{19} = e_{12} - e_5 e_{14}/e_1 - e_9 e_{16}/e_7 + e_3 e_9 e_{14}/(e_1 e_7)$$

$$e_{20} = e_{13} - e_{10} e_{16}/e_7 + e_3 e_{10} e_{14}/(e_1 e_7)$$

$$e_{21} = e_{15} - e_2 e_{14}/e_1 - e_6 e_{16}/e_7 + e_3 e_6 e_{14}/(e_1 e_7)$$

$$e_{22} = e_{17} - e_8 e_{16}/e_7 + e_3 e_8 e_{14}/(e_1 e_7).$$

APPENDIX II

The coefficients g_i.

HSDT

$$g_1 = (e_{10} f_1 - e_3 f_4)/e_0,\quad g_2 = (e_{12} f_5 - e_{19} f_2)/c_0,$$

$$g_4 = (e_1 f_4 - e_{28} f_1)/e_0,\quad g_5 = (e_{17} f_2 - e_{10} f_5)/c_0,$$

$$g_3 = a_0(g_2 a_1 + g_5 a_2 - f_3 + \alpha e_{31} g_1 + \alpha e_{33} g_4)$$

where

$$f_1 = \alpha L_1 \bar{T}_0 + \alpha L_2 \bar{T}_1$$

$$f_2 = \beta L_3 \bar{T}_0 + \beta L_4 \bar{T}_1$$

$$f_3 = -c_2(\alpha^2 L_7 + \beta^2 L_9)\bar{T}_0 - \left(\frac{L_1}{R_1} + \frac{L_3}{R_2}\right)\bar{T}_0 - c_2(\alpha^2 L_8 + \beta^2 L_{10})\bar{T}_1 - \left(\frac{L_2}{R_1} + \frac{L_4}{R_2}\right)\bar{T}_1$$

$$f_4 = (\alpha L_2 - c_2 \alpha L_7)\bar{T}_0 + (\alpha L_5 - \alpha c_2 L_8)\bar{T}_1$$

$$f_5 = (\beta L_4 - c_2 \beta L_9)\bar{T}_0 + (\beta L_6 - \beta c_2 L_{10})\bar{T}_1$$

FSDT

$$g_1 = (e_3 f_4 - e_{19} f_1)/e_0,\quad g_2 = (e_{10} f_5 - e_{27} f_2)/c_0,\quad g_3 = f_3/e_{13}$$

$$g_4 = (e_{17} f_1 - e_1 f_4)/e_0,\quad g_5 = (e_{24} f_2 - e_7 f_5)/c_0$$

where

$$f_1 = \alpha L_1 \bar{T}_0 + \alpha L_2 \bar{T}_1$$

$$f_2 = \beta L_3 \bar{T}_0 + \beta L_4 \bar{T}_1$$

$$f_3 = -\left(\frac{L_1}{R_1} + \frac{L_3}{R_2}\right)\bar{T}_0 - \left(\frac{L_2}{R_1} + \frac{L_4}{R_2}\right)\bar{T}_1$$

$$f_4 = \alpha L_2 \bar{T}_0 + \alpha L_5 \bar{T}_1$$

$$f_5 = \beta L_4 \bar{T}_0 + \beta L_6 \bar{T}_1$$

CST

$$g_1 = -\frac{f_1}{e_1},\quad g_2 = \frac{f_2}{e_7},$$

$$g_3 = \alpha\frac{e_{14}f_1}{e_1e_{18}} - \frac{e_{16}f_2}{e_7e_{18}} + \frac{e_3e_{14}f_2}{e_1e_7e_{18}} + \frac{f_3}{e_{18}}$$

where

$$f_1 = \alpha L_1\bar{T}_0 + \alpha L_2\bar{T}_1,$$

$$f_2 = \beta L_3\bar{T}_0 + \beta L_4\bar{T}_1,$$

$$f_3 = \left(\alpha^2L_2 + \beta^2L_4 + \frac{L_1}{R_1} + \frac{L_3}{R_2}\right)\bar{T}_0 + \left(\alpha^2L_5 + \beta^2L_6 + \frac{L_2}{R_1} + \frac{L_4}{R_2}\right)\bar{T}_1$$

and

$$\begin{Bmatrix} L_1, & L_2, & L_5, & L_7, & L_8 \\ L_3, & L_4, & L_6, & L_9, & L_{10} \end{Bmatrix} = \sum_{k=1}^{N}\int_{\zeta_{k-1}}^{\zeta_k}\begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{12}^{(k)} & Q_{22}^{(k)} \end{bmatrix}\begin{Bmatrix} \alpha_{11}^{(k)} \\ \alpha_{22}^{(k)} \end{Bmatrix}(1, \zeta, \zeta^2, \zeta^3, \zeta^4)\,\mathrm{d}\zeta.$$